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Shape Invariant Potential and Semi-Unitary Transformations (SUT) for Supersymmetry Harmonic Oscillator in T^4-Space

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Abstract Constructing the Semi-Unitary Transformations (SUT) to obtain the supersymmetric partner Hamiltonians for a one dimensional harmonic oscillator, it has been shown that under this transformation the supersymmetric partner loses its ground state in T^4-space while its eigen functions constitute a complete orthonormal basis in a subspace of full Hilbert space.

Keywords Supersymmetry · Superluminal Transformations · Semi Unitary Transformations

There are a number of analytically solvable problems in non-relativistic quantum mechanics for which all the energy eigen values and eigen functions are explicably known. The question naturally arises as to why these potentials are solvable and what is the underlying symmetry property? No definite answer was known until 1983 when in a largely unnoticed paper, Grenden Shtein [1] pointed out that all these potentials have a property of shape invariance. Keeping in view the recent potential importance of tachyons [2–13] and the fact that these particles are not contradictory to special theory of relativity and are localized in time in view of second quantization and interaction of superluminal electromagnetic fields [14–17], we have constructed a Semi-Unitary Transformation (SUT) to obtain the supersymmetric partner Hamiltonians for one dimensional harmonic oscillator in T^4-space (i.e the localization space for tachyons [18–21]) and it has been demonstrated that under this SUT the supersymmetric partner Hamiltonian H_+^T loses its ground state while its eigen functions constitute a complete orthonormal set in a subspace of full Hilbert space.

In order to over come the various problems associated with superluminal Lorentz transformations (SLTs) [4–13], six-dimensional formalism [22–26] of space time is adopted with the symmetric structure of space and time having three space and three time components

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of a six dimensional space time vector. In this formalism, a subluminal observer $\mathcal{O}$ in the usual $R^4 \equiv (\vec{r}, t)$ space is surrounded by a neighborhood in which one measures the scalar time $|t| \equiv |(t_x^2 + t_y^2 + t_z^2)|^{\frac{1}{2}}$ and spatial vector $\vec{r} = (x, y, z)$ out of six independent coordinates (x, y, z, t_x, t_y, t_z) of the six-dimensional space R^6. On passing from $R^6 = (\vec{r}, \vec{t})$ to $(R^6)' = (\vec{r}', \vec{t}')$ via SLT's, the usual $(R^4)' = (\vec{r}', t')$ of observer $\mathcal{O}'$ in R^6 will appear as $T^4 \equiv (t_x', t_y', t_z', r')$ to the observer $\mathcal{O}$ in R^6. The resulting space for bradyons and tachyons is thus identified as the R^6- or $M(3,3)$ space where both space and time and hence energy and momentum are considered as vector quantities. Superluminal Lorentz transformations (SLTs) between two frames K and K' moving with velocity $v > 1$ (in the natural units $c = \hbar = 1$) are defined in R^6- or $M(3,3)$ space as follows

$$
\begin{aligned}
x' &\to \pm t_x,\\
y' &\to \pm t_y,\\
z' &\to \pm\gamma(z - vt_z),\\
t_x' &\to \pm x,\\
t_y' &\to \pm y,\\
t_z' &\to \pm\gamma(t_z - vz),
\end{aligned}
\tag{1}
$$

where $\gamma = (v^2 - 1)^{\frac{1}{2}}$. The above transformations lead to the mixing of space and time coordinates for transcendental tachyonic objects, $(|\vec{v}| \to \infty)$ where (1) takes the following form

$$
\begin{aligned}
+dt_x &\to dt_x' = dx+,\\
+dt_y &\to dt_y' = dy+,\\
+dt_z &\to dt_z' = dz+,\\
-dz &\to dz' = dt_z-,\\
-dy &\to dy' = dt_y-,\\
-dx &\to dx' = dt_x - .
\end{aligned}
\tag{2}
$$

The above equation shows that we have only two four dimensional slices of R^6- or $M(3,3)$ space $(+,+,+,-)$ and $(-,-,-,+)$. When any reference frame describes bradyonic objects it is necessary to describe $M(1,3) = [t, x, y, z]$ (R^4-space) so that the coordinates t_x and t_y are not observed or couple together giving $t = |\vec{t}| \equiv |(t_x^2 + t_y^2 + t_z^2)|^{\frac{1}{2}}$. On the other hand when a frame describes bradyonic object in frame K, it will describe a tachyonic object (with velocity $(|\vec{v}| \to \infty)$ in K' with $M'(1,3)$ space i.e. $M'(1,3) = [(t_x', x', y', z'] = [z, t_x, t_y, t_z]$ (T^4-space). We define $M'(1,3)$ space as T^4-space or $M(3,1)$ space where x and y are not observed or coupled together giving rise to $|\vec{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$. As such, the spaces R^4 and T^4 are two observational slices of R^6- or $M(3,3)$ space but unfortunately the space is not consistent with special theory of relativity. Subluminal and superluminal Lorentz transformations lose their meaning in R^6- or $M(3,3)$ space with the sense that these transformations do not represent either the bradyonic or tachyonic objects in this space. It has been shown earlier [18–21] that the true localizations space for bradyons is R^4-space while that for tachyons is T^4- space. So a bradyonic $R^4 = M(1,3)$ space maps to a tachyonic $T^4 = M'(3, 1)$ space or vice versa i.e.

$$R^4 = M(1, 3) \overset{SLT}{\rightarrow} M'(3, 1) = T^4. \tag{3}$$

Let us describe the supersymmetric quantum mechanics [27] in terms of a pair of bosonic Hamiltonians H_+^T and H_-^T which are supersymmetric partners [28] of supersymmetric Hamiltonian for tachyons in T^4-space i.e.,

$$H^T = H_+^T \oplus H_+^T. \tag{4}$$

In order to construct these super partner Hamiltonians for the system described by $H^T = H_B^T \oplus H_F^T$, we may introduce the potential $V_-(t)$ whose ground state energy has been adjusted to zero with the corresponding ground state wave function $\psi_0^{T(-)}$ given by

$$\psi_0^{T(-)} = \exp\left[-\int_0^t W(t')dt'\right]. \tag{5}$$

Substituting it in the Schrödinger equation (in the units of $\hbar = 2k = 1$), we get

$$V_-(t) = W^2(t) - W'(t) = \frac{\psi_0^{T''(-)}}{\psi_0^{(-)}}. \tag{6}$$

Correspondingly for the potential, we have the following Hamiltonian

$$H_-^T = -\frac{d^2}{dt^2} + \frac{\psi_0^{T''(-)}}{\psi_0^{(-)}} = -\frac{d^2}{dt^2} + V_-(t). \tag{7}$$

If the ground state wave function $\psi_0^{T(-)}$ is square integrable then the supersymmetry will be considered to be broken. This Hamiltonian may also be written in the following form in terms of bosonic operator $\hat{B}$ and $\hat{B}^+$;

$$H_-^T = \hat{B}^+\hat{B} \tag{8}$$

where

$$\hat{B} = \frac{d}{dt} + W(t) = \frac{d}{dt} - \frac{\psi_0^{T''(-)}}{\psi_0^{(-)}}; \tag{9}$$

$$\hat{B}^+ = -\frac{d}{dt} + W(t) = -\frac{d}{dt} - \frac{\psi_0^{T''(-)}}{\psi_0^{(-)}}. \tag{10}$$

Let us introduce the Hamiltonian

$$H_+^T = \hat{B}\hat{B}^+ = -\frac{d^2}{dt^2} + V_+(t) = -\frac{d^2}{dt^2} + W^2(t) + W'(t) \tag{11}$$

where

$$V_+(t) = W^2(t) + W'(t) = V_-(t) + 2W'(t). \tag{12}$$

The potentials $V_+(t)$ and $V_-(t)$ are called supersymmetric partner potentials and H_+ is the Hamiltonian corresponding to the potential $V_+(t)$. We may now impose the following condition on the supersymmetry partner potentials $V_+(t)$ and $V_-(t)$ as

$$V_+(t;c_0) = V_-(t;c_1) + R(c_1) \tag{13}$$

where c_0 is a set of parameters occurring in $V_+(t)$ and c_1 is a function of c_0 while the remainder $R(c_1)$ is independent of t. Then the supersymmetric partner potentials $V_+(t)$ and $V_+(t)$ are said to be shape invariant. All potentials, which exhibit the property of shape invariance, can exactly be solved. It can be demonstrated in the straightforward manner by constructing the sequence $H^{T(k)}$ of the Hamiltonians as

$$H^{T(k)} = -\frac{d^2}{dt^2} + V_-(t;c_k) + \sum_{i=1}^{k} R(c_i) \tag{14}$$

where $c_1 = f(c_0)$ and $c_i = f^i(c_0) = f(c_0)\cdots i$ times. It is obvious that

$$H^{T(0)} = H^{T(1)}, \qquad H_-^T = H_+^T. \tag{15}$$

For all $k > 0$, $H^{T(k)}$ and $H^{T(k+1)}$ are supersymmetry partner Hamiltonians since they have identical bound state spectra except for the lowest level of $H^{T(k)}$ i.e.

$$E_0^{T(k)} = \sum_{i=1}^{k} R(c_i). \tag{16}$$

Furthermore, the ground state energy of $H^{T(k+1)}$ coincides with the first excited state energy of $H^{T(k)}$ for all $k > 0$. Thus we have ground state energy $H^{T(n)}$ as the nth state energy of $H^{T(0)}$. Combining this result with (16), we get

$$E_0^{T(k)} = \sum_{i=1}^{k} R(c_i), \qquad E_0^{T(-)} = 0. \tag{17}$$

For the Harmonic oscillator, we may define

$$V_+(t) = V_-(t) + 2\Omega. \tag{18}$$

Here Ω is considered as the part of one-dimensional supersymmetric harmonic oscillator $W(t) = \Omega t$ and with $H_-^T = \frac{d^2}{dt^2} + \Omega^2 t^2 - \Omega$ with $V_-(t) = \Omega^2 t^2 - \Omega$. Comparing (18) with (13), we get

$$c_0 = c_1 = \Omega, \qquad R(c_1) = 2\Omega. \tag{19}$$

Then (14) becomes

$$H^{T(k)} = \frac{d^2}{dt^2} + \Omega^2 t^2 + (2k-1)\Omega \tag{20}$$

which reproduces the results (15). Equation (20) may also be written as

$$H^{T(k+1)} = \frac{d^2}{dt^2} + \Omega^2 t^2 + (2k+1)\Omega = H^{T(k)} + \Omega'. \tag{21}$$

Then (16) gives

$$E_0^{T(n)} = n\Omega' \tag{22}$$

showing that the ground state energy of $H^{T(n)}$ is identical with the nth state energy of $H^{T(0)} = H_-^T$. In the similar manner, using the shape invariance property of partner potentials $V_+(t)$ and $V_-(t)$, since for shape invariant potential we may reproduce the other results of supersymmetric harmonic oscillator

$$\psi_n^{T(+)}(t; c_0) = \psi_n^{T(-)}(t; c_1). \tag{23}$$

The method of shape invariant potentials used here for harmonic oscillator can thus be generalized to all shape invariant potentials for the deeper understanding of analytically solvable potentials. It is obvious from (21) that operator $H_+^T = BB^+$ is positive definite for all states while the operator $H_-^T = B^+B$ is semi-positive definite since $(B^+B)^{-1}$ is singular for $n = 0$ in (22). Super partner Hamiltonians H_-^T and H_+^T are obviously Hermitian. Thus we can construct the following operators

$$U = H_+^{T(-\frac{1}{2})} B, \qquad U^+ = B^+ H_+^{T(-\frac{1}{2})} \tag{24}$$

which gives

$$UU^+ = I, \qquad U^+U = B^+ H_+^{T(-)} B = P \neq I \tag{25}$$

showing that the operator U is semi unitary and hence any transformation involving U will be semi unitary transformation (SUT). The operator P defined by (25) satisfies the following conditions

$$P^2 = P, \qquad P^+ = P \tag{26}$$

showing that it is a projection operator having the eigen values 0 and 1. Under the SUT defined by (24), we have

$$U H_-^T U^+ = H_+^T. \tag{27}$$

We also have

$$[P, H_-^T] = 0 \tag{28}$$

which shows that H_-^T and P have common eigen functions and eigen values of P are good quantum members in T^4-space. Using equation

$$\begin{aligned} &\hat{B}^+\hat{B}\psi^{T(-)} = E^T\psi^{T(-)}, \qquad \hat{B}\hat{B}^+(\hat{B}\psi^{T(-)}) = E^T\hat{B}\psi^{T(-)}, \\ &H_+^T[\hat{B}\psi^{T(-)}] = E^T[\hat{B}\psi^{T(-)}] \end{aligned} \tag{29}$$

and $E_n^{T(+)} = E_{n+1}^{T(-)}$, we get

$$P\psi_n^{T(-)} = \psi_n^{T(-)} \quad \text{for } n > 0, \qquad P\psi_0^{T(-)} = 0. \tag{30}$$

In general we know

$$P\psi_{n+1}^{T(-)} = \psi_{n+1}^{T(-)}. \tag{31}$$

Let us denote

$$\psi_0^{T(-)} = |0,0\rangle, \qquad \psi_{n+1}^{T(-)} = |n+1,1\rangle \tag{32}$$

as the eigen state for harmonic oscillator, we may readily obtain,

$$\begin{aligned} H_-^T|n+1,1\rangle &= (n+1)\Omega|n+1,1\rangle, \\ P|n+1,1\rangle &= |n+1,1\rangle, \\ H_-^T|0,0\rangle &= 0, \\ P|0,0\rangle &= 0. \end{aligned} \tag{33}$$

It shows that under the projection P the full Hilbert space H of harmonic oscillator is projected in two subspaces

$$H = H_0 \oplus H_1 \tag{34}$$

where the subspace H_0 is constituted by the state $|0,0\rangle$ and the subspace H_1 is constituted by the states $|n+1,1\rangle$. To find the general structure of the operator P in the above basis, let us start with the complete set $|n\rangle$ of H_-^T (for $n = 0, 1, 2, \ldots$), and construct

$$|n\rangle_+ = U^+|n\rangle = H_+^{T(-\frac{1}{2})} B|n\rangle \tag{35}$$

which gives

$$\sum_{n=0}^{\infty} |n\rangle_{++}\langle n| = U^+ \sum_{n=0}^{\infty} |n\rangle\langle n| U = U^+U = P \tag{36}$$

showing that states $|n\rangle_+$ do not form the complete set. We obviously have

$$P|n\rangle_+ = |n\rangle_+, \qquad |n\rangle_+ = |P=1\rangle. \tag{37}$$

The complete set of the eigen states is thus formed by the states $|q=0\rangle$ and $|n\rangle_+$ i.e.

$$|q=0\rangle\langle q=0| + \sum_n |n\rangle_{++}\langle n| = I, \qquad \sum_n |n\rangle_{++}\langle n| = I - |0\rangle\langle 0|. \tag{38}$$

As such, the matrix of the operator P is diagonal in the basis given by (31) with the general structure

$$P = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} = I - |0\rangle\langle 0|. \tag{39}$$

Furthermore, from SUT transformation (35) we get

$$_+\langle n, n\rangle_+ = \langle n|UU^+|n\rangle = \langle n, n\rangle \tag{40}$$

showing that the orthonormality of the states $|n\rangle$ implies the orthonormality of states $|n\rangle_+$. Thus (38) and (40) demonstrate that under the SUT transformation (35) the orthogonality and the normalization of states are maintained while the completeness condition is violated. On the other hand, let us consider the SUT transformation

$$|n\rangle_- = U|n\rangle \tag{41}$$

where $|n\rangle$ constitute the complete orthonormal set of states of H_-^T. Then we have

$$\sum_{n=0}^{\infty} |n\rangle_{--}\langle n| = \sum_{n=0}^{\infty} U|n\rangle\langle n|U^+ = UU^+ = I \tag{42}$$

showing that the completeness of set $|n\rangle$ implies the completeness of set $|n\rangle_-$. But

$$_-\langle m, n\rangle_- = \langle m|U^+U|n\rangle = \langle m|P|n\rangle \neq \delta_{mn}. \tag{43}$$

In other words the transformation (41) maintains completeness relation but fails to maintain orthogonality and normalization conditions. However, under the transformation (35) we have

$$|0, 0\rangle_- = U|0, 0\rangle = 0 \tag{44}$$

which shows that this SUT destroys the ground state $|0, 0\rangle$, while the state

$$|n + 1, 1\rangle_- = U|n\rangle \tag{45}$$

satisfies the following conditions by using relation (44)

$$\sum_n |n + 1, 1\rangle_{--}\langle n + 1, 1| = UU^+ = I; \tag{46}$$

$$\begin{aligned} _-\langle m + 1, 1|n + 1, 1\rangle_- &= \langle m + 1|P|n + 1\rangle = \langle m + 1|I - 0, 0\rangle\langle 0, 0|n + 1\rangle \\ &= \langle m + 1|n + 1\rangle = \delta_{mn}. \end{aligned} \tag{47}$$

Thus the orthogonality and normalization conditions are restored for the states $|n + 1, 1\rangle_-$, which constitute the basis of H_1. In other words, the states $|n + 1, 1\rangle_+$ constitute complete orthonormal set, even though $|0, 0\rangle_-$, is destroyed. It shows that the supersymmetric partner H_+^T, compared with H_-^T, loses the ground state but its eigen function $\psi_n^{T(+)} = U\psi_{n+1}^{T(-)}$ with eigen values $E_n^{T(+)} = E_{n+1}^{T(-)}$ constitutes a complete orthonormal set in the subspace H_1. Thus the SUT introduced here provides a new way to relate solvable systems to their supersymmetric partners and helps us to construct a new class of solvable potentials when we start from solvable systems where Hamiltonian can be factorized. It has shown that semi unitary transformations operators U and U^+ constructed above describes a projection operator having eigen values 0 and 1 while the commutation relation shows that H_-^T and P have common eigen functions and eigen values in T^4-space. It has been discussed that under the projection P the full Hilbert space of harmonic oscillator is decomposed in two subspaces and the states $|n\rangle_+$ associated with semi unitary operator in T^4-space form the complete set. The diagonalization of projection operator P has been illustrated and accordingly the orthonormality condition leads to the conclusion that the orthonormality of state $|n\rangle$ implies the orthonormality of states $|n\rangle_+$. As such it is claimed that under the SUT transformation the orthogonality and normalization of states are maintained while the completeness condition has been said to be violated. It has also been shown that the orthogonality and normalization conditions are restored for the states $|n+1,1\rangle_-$. In other words the states $|n+1,1\rangle_-$ constitute complete orthonormal set, even though $|0,0\rangle_-$ is destroyed. It shows that supersymmetric partner H_+^T, compared with H_-^T, loses the ground state but its eigen function $\psi_n^{T(+)} = U\psi_{n+1}^{T(-)}$ with eigen value $E_n^{T(+)} = E_{n+1}^{T(-)}$ constitute a complete orthonormal set. It has already been emphasized earlier [18–21] that on passing from bradyons to tachyons, the role of space and time and consequently momentum and energy are changed. Thus the positivity of energy for bradyonic particles is considered only in four-dimensional space with three space and one time coordinates i.e. in R^4-space while for the case of tachyons the Hamiltonian is space dependent in T^4-space and the positivity of momentum is being taken in to account. The T^4-space has been visualized as the space for tachyons where they behave as bradyons do in R^4-space. The fore going analysis for tachyons and their behavior in supersymmetric quantum mechanics leads to the conclusion that there has been basic disagreement in localization and representation of tachyons. As such the observables of more than four dimensions of space-time may be associated with the even horizon effects ($R^4 \to T^4$) taking into account the interconnection between boson-fermion symmetry and bradyon tachyon transformations. The semi unitary transformation (SUT) introduced here provides a new window to relate solvable system to their supersymmetric partners and helps us to construct a new class of solvable potentials when we start from solvable system whose Hamiltonian can be factorized on passing from bradyons to tachyons in T^4-space.

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